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A Cartesian-tensor solution of the Brinkman equation

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Abstract. A method of solving the Brinkman equation, using Cartesian tensors, is developed. General expressions for the velocity vector and the pressure, which can directly be used in situations where boundary conditions are expressible in Cartesian-tensor form, are obtained. It is shown how the drag on a porous sphere can be directly and easily calculated using this method.

1. Introduction

There are very few good model equations which give a faithful macroscopic description of the flow in a porous medium. Basically, there have been differences of opinion as to which is the most appropriate form of the equation of motion. While everyone recognizes the limitations of Darcy's law, there is no other equation in the literature which has been unconditionally accepted by different authors. Thus there have been proposals to add non-linear inertia terms, quadratic velocity terms, etc. in the equations of motion or to employ Darcy's law but to allow discontinuities in the fluid velocity at the surface of the medium.

One equation which has gained considerable success, particularly when the motion is slow, is the Brinkman equation [1]. It is believed that when the porosity is large, this equation, which is obtained by adding a Laplacian term in velocity to Darcy's law, gives satisfactory results. By considering slow flow in random arrays of fixed spheres and for suspensions, Howells [2] and Hinch [3], respectively, confirm the validity of the Brinkman equation. There is also the experimental verification of this equation in the works of Matsumoto and Suganuma [4], who measured the settling velocity of model flocs made of steel wool.

In the present note we develop a general solution of the Brinkman equation, based upon the use of Cartesian tensors. In essence it is an extension of the method developed earlier [5] to study the viscous creeping-motion equation. An integral-equation approach, making use of Green's function for the Brinkman equation, has been discussed by Higdon and Kojima [6]. It is believed that the solution obtained by the present method will be directly applicable to porous-flow problems for which the boundary conditions are given in the Cartesian-tensor form and to those that can easily be written in this manner. In the next section we develop the solution based upon the use of arbitrary, spatially constant, second- and third-order tensors. In this manner we are able to determine the general expressions for the velocity and pressure field. To illustrate the use of the method, we apply it to find the drag of a porous sphere in a single-fluid flow.

2. Basic equations and their solution

We consider the velocity field satisfying the continuity equation

$$\nabla \cdot \hat{\mathbf{u}} = 0, \tag{1}$$

and the Brinkman equation

$$-\nabla \hat{p} + \mu \nabla^2 \hat{\mathbf{u}} = \frac{\mu}{k} \hat{\mathbf{u}}. \tag{2}$$

Here $\hat{\mathbf{u}}$ is the volume average velocity, $\hat{p}$ the interstitial average pressure, μ the viscosity of the fluid and k the permeability of the porous medium. The above are four partial differential equations for the four unknowns $\hat{\mathbf{u}}$ and $\hat{p}$. On taking the divergence of equation (2) and using (1), it turns out that solving the above system is equivalent to finding the solution of the equations

$$\nabla^2 p = 0, \tag{3}$$

$$\nabla^2 \hat{\mathbf{u}} - C^2 \hat{\mathbf{u}} = \nabla p \tag{4}$$

where $p = \mu^{-1} \hat{p}$ and $C^2 = k^{-1}$. Once $\hat{\mathbf{u}}$ is so determined we require it to satisfy equation (1).

We now generate solutions of the above system of equations which involve spatially-constant second- and third-order tensors a_{ij} and b_{ijk}, respectively. Following [5], we write the scalar invariants linear in a_{ij} and b_{ijk}, in combination with the position vector x_i ($r^2 = x_i x_i$), as

$$a_{ii},\ \varepsilon_{ijk} a_{kj} x_i,\ a_{ij} x_i x_j,\ b_{imm} x_i,\ b_{mim} x_i,\ b_{mmi} x_i,\ b_{ijk} x_i x_j x_k. \tag{5}$$

On assuming the form of $p(r)$ to be

$$p(r) = H^0(r) a_{ii} + H^1(r) \varepsilon_{ijk} a_{jk} x_i + H^2(r) b_{imm} x_i + H^3(r) b_{mim} x_i + H^4(r) b_{mmi} x_i$$
$$+ H^5(r) a_{ij} x_i x_j + H^6(r) b_{ijk} x_i x_j x_k, \tag{6}$$

it is found from (3) that (cf. [5])

$$H^0(r) = -\tfrac{1}{3} B_1^5 r^{-3} + B_1^0 r^{-1} + A_1^0 - \tfrac{1}{3} A^5 r^2,$$

$$H^1(r) = B_1^1 r^{-3} + A_1^1,$$

$$H^2(r) = -\tfrac{1}{5} B_1^6 r^{-5} + B_1^2 r^{-3} + A_2^2 - \tfrac{1}{5} A_1^6 r^2,$$

$$H^3(r) = -\tfrac{1}{5} B_1^6 r^{-5} + B_1^3 r^{-3} + A_2^3 - \tfrac{1}{5} A_1^6 r^2,$$

$$H^4(r) = -\tfrac{1}{5}B_1^6 r^{-5} + B_1^4 r^{-3} + A_2^4 - \tfrac{1}{5}A_1^6 r^2,$$

$$H^5(r) = B_1^5 r^{-5} + A_1^5,$$

$$H^6(r) = B_1^6 r^{-7} + A_1^6, \tag{7}$$

where A_j^i and B_j^i are abitrary constants.

Equations (4) and (6) suggest that the velocity component of $\hat{\mathbf{u}}$ in the direction of x_p can be assumed to be of the form

$$\begin{aligned}\hat{u}_p(r) = {} & h_1^0(r)a_{ii}x_p + h_1^1(r)\varepsilon_{ijk}a_{kj}x_i x_p + h_2^1(r)\varepsilon_{pjk}a_{kj} + h_1^2(r)b_{imm}x_i x_p + h_2^2(r)b_{pmm} \\ & + h_1^3(r)b_{mim}x_i x_p + h_2^3(r)b_{mpm} + h_1^4(r)b_{mmi}x_i x_p + h_2^4(r)b_{mmp} + h_1^5(r)a_{ij}x_i x_j x_p \\ & + h_2^5(r)a_{pj}x_j + h_3^5(r)a_{jp}x_j + h_1^6(r)b_{ijk}x_i x_j x_k x_p + h_2^6(r)b_{pjk}x_j x_k + h_3^6(r)b_{jpk}x_j x_k \\ & + h_4^6(r)b_{jkp}x_j x_k.\end{aligned} \tag{8}$$

Substitution of (8) and (6) in equation (4) yields the following type of differential equations:

$$(h_1^0)'' + \frac{4}{r}(h_1^0)' - C^2 h_1^0 = \frac{1}{r}(H^0)' - 2h_1^5,$$

$$(h_1^1)'' + \frac{6}{r}(h_1^1)' - C^2 h_1^1 = \frac{1}{r}(H^1)',$$

$$(h_2^1)'' + \frac{2}{r}(h_2^1)' - C^2 h_2^1 = H^1 - 2h_1^1,$$

$$(h_2^2)'' + \frac{2}{r}(h_2^2)' - C^2 h_2^2 = H^2 - 2h_1^2 - 2h_1^6,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(h_4^6)'' + \frac{6}{r}(h_4^6)' - C^2 h_4^6 = H^6 - 2h_1^6, \tag{9}$$

where the dashes denote differentiation with respect to the variable r.

We now consider the following equation,

$$y''(r) + \frac{2l}{r}y'(r) - C^2 y(r) = 0 \ (l = 1, 2, 3, 4, 5), \tag{10}$$

and state some properties of the solutions of (10) which are used to construct the solutions of equations (9).

PROPERTY 1. If $y_l(r)$ satisfies (10) then $y_{l+1}(r) = (1/r)(y_l)'$ satisfies

$$y'' + \frac{2(l+1)}{r} y' - C^2 y = 0. \tag{11}$$

PROPERTY 2. If $y_{l+1}(r)$ satisfies equation (11), then $y_{l+1}(r)$ also satisfies the inhomogeneous equations

$$y'' + \frac{2l}{r} y' - C^2 y = -2y_{l+2} \tag{12}$$

and

$$y'' + \frac{2(l-1)}{r} y' - C^2 y = -4y_{l+2}, \quad l \geqslant 2. \tag{13}$$

The proofs of the above properties can be verified by substitution of the appropriate expressions in the above equations and rearranging the resulting terms.

Let $\varrho = Cr$ and $y = u\varrho^{-l+1/2}$. Equation (10) then becomes a modified Bessel equation,

$$\varrho^2 \frac{d^2 u}{d\varrho^2} + \varrho \frac{du}{d\varrho} - [\varrho^2 + (l - \tfrac{1}{2})^2]\, u = 0, \tag{14}$$

and admits $I_{l-1/2}(\varrho)$ and $I_{-l+1/2}(\varrho)$ as two linearly independent solutions. Therefore, it follows that

$$\begin{aligned} y_l &= C^{2l-1}(Cr)^{-l+1/2} I_{l-1/2}(Cr), \\ y_l^* &= C^{2l-1}(Cr)^{-l+1/2} I_{-l+1/2}(Cr) \end{aligned} \tag{15}$$

are two linearly-independent solutions of (10).

PROPERTY 3. For y_l and y_l^* as defined in (15), we have

$$y_{l+1} = \frac{1}{r}\frac{d}{dr} y_l(r), \quad y_{l+1}^* = \frac{1}{r}\frac{d}{dr} y_l^*(r), \tag{16}$$

and

$$\begin{aligned} r\frac{d}{dr} y_{l+1}(r) + (2l+1) y_{l+1}(r) &= C^2 y_l(r), \\ r\frac{d}{dr} y_{l+1}^*(r) + (2l+1) y_{l+1}^*(r) &= C^2 y_l^*(r). \end{aligned} \tag{17}$$

The verification of the above, by use of the properties of Bessel functions, is straightforward.

Making use of the above properties, we now write down the solutions of equations (9). These are:

$$
\begin{aligned}
h_1^0(r) &= -\frac{1}{C^2} B_1^5 r^{-5} + \frac{1}{C^2} B_1^0 r^{-3} + \frac{2}{3C^2} A_1^5 + A_2^5 y_3(r) + A_2^0 y_2(r) + B_2^5 y_3^*(r) \\
&\quad + B_2^0 y_2^*(r), \\
h_1^1(r) &= \frac{3}{C^2} B_1^1 r^{-5} + A_2^1 y_3(r) + B_2^1 y_3^*(r), \\
h_2^1(r) &= -\frac{1}{C^2} B_1^1 r^{-3} - \frac{1}{C^2} A_1^1 + A_2^1 y_2(r) + A_3^1 y_1(r) + B_2^1 y_2^*(r) + B_3^1 y_1^*(r), \\
h_1^2(r) &= -\frac{1}{C^2} B_1^6 r^{-7} + \frac{3}{C^2} B_1^2 r^{-5} + \frac{2}{5C^2} A_1^6 + A_2^6 y_4(r) + A_3^2 y_3(r) \\
&\quad + B_2^6 y_4^*(r) + B_3^2 y_3^*(r), \\
h_2^2(r) &= \frac{1}{5C^2} B_1^6 r^{-5} - \frac{1}{C^2} B_1^2 r^{-3} - \frac{1}{C^2} A_2^2 + \frac{1}{5C^2} A_1^6 r^2 + A_2^6 y_3(r) + (A_3^2 + A_3^6) y_2(r) \\
&\quad + A_4^2 y_1(r) + B_2^6 y_3^*(r) + (B_3^2 + B_3^6) y_2^*(r) + B_4^2 y_1^*(r), \\
h_1^3(r) &= -\frac{1}{C^2} B_1^6 r^{-7} + \frac{1}{C^2} B_1^3 r^{-5} + \frac{2}{5C^2} A_1^6 + A_2^6 y_4(r) + A_3^3 y_3(r) + B_2^6 y_4^*(r) \\
&\quad + B_3^3 y_3^*(r), \\
h_2^3(r) &= \frac{1}{5C^2} B_1^6 r^{-5} - \frac{3}{C^2} B_1^3 r^{-3} - \frac{1}{C^2} A_2^3 + \frac{1}{5C^2} A_1^6 r^2 + A_2^6 y_3(r) + (A_3^3 + A_4^6) y_2(r) \\
&\quad + A_4^3 y_1(r) + B_2^6 y_3^*(r) + (B_3^3 + B_4^6) y_2^*(r) + B_4^3 y_1^*(r), \\
h_1^4(r) &= \frac{1}{C^2} B_1^6 r^{-7} + \frac{3}{C^2} B_1^4 r^{-5} + \frac{2}{5C^2} A_1^6 + A_2^6 y_4(r) + A_3^4 y_3(r) + B_2^6 y_4^*(r) + B_3^4 y_3^*(r), \\
h_2^4(r) &= \frac{1}{5C^2} B_1^6 r^{-5} - \frac{1}{C^2} B_1^4 r^{-3} - \frac{1}{C^2} A_2^4 + \frac{1}{5C^2} A_1^6 r^2 + A_2^6 y_3(r) + (A_3^4 + A_5^6) y_2(r) \\
&\quad + A_4^4 y_1(r) + B_2^6 y_3^*(r) + (B_3^4 + B_5^6) y_2^*(r) + B_4^4 y_1^*(r), \\
h_1^5(r) &= \frac{5}{C^2} B_1^5 r^{-7} + A_2^5 y_4(r) + B_2^5 y_4^*(r),
\end{aligned}
$$

$$h_2^5(r) = -\frac{1}{C^2}B_1^5 r^{-5} - \frac{1}{C^2}A_1^5 + A_2^5 y_3(r) + A_3^5 y_2(r) + B_2^5 y_3^*(r) + B_3^5 y_2^*(r),$$

$$h_3^5(r) = -\frac{1}{C^2}B_1^5 r^{-5} - \frac{1}{C^2}A_1^5 + A_2^5 y_3(r) + A_4^5 y_2(r) + B_2^5 y_3^*(r) + B_4^5 y_2^*(r),$$

$$h_1^6(r) = \frac{7}{C^2}B_1^6 r^{-9} + A_2^6 y_5(r) + B_2^6 y_5^*(r),$$

$$h_2^6(r) = -\frac{1}{C^2}B_1^6 r^{-7} - \frac{1}{C^2}A_1^6 + A_2^6 y_4(r) + A_3^6 y_3(r) + B_2^6 y_4{}^*(r) + B_3^6 y_3^*(r),$$

$$h_3^6(r) = -\frac{1}{C^2}B_1^6 r^{-7} - \frac{1}{C^2}A_1^6 + A_2^6 y_4(r) + A_4^6 y_3(r) + B_2^6 y_4^*(r) + B_4^6 y_3^*(r),$$

$$h_4^6(r) = -\frac{1}{C^2}B_1^6 r^{-7} - \frac{1}{C^2}A_1^6 + A_2^6 y_4(r) + A_5^6 y_3(r) + B_2^6 y_4^*(r) + B_5^6 y_3^*(r). \tag{18}$$

The equation of continuity imposes the following restrictions upon the arbitrary constants:

$$A_2^0 = 0, \qquad B_2^0 = 0,$$

$$A_2^5 = -\frac{A_3^5 + A_4^5}{C^2}, \qquad B_2^5 = -\frac{B_3^5 + B_4^5}{C^2},$$

$$A_2^1 = -\frac{A_3^1}{C^2}, \qquad B_2^1 = -\frac{B_3^1}{C^2},$$

$$A_2^6 = -\frac{A_3^6 + A_4^6 + A_5^6}{C^2}, \qquad B_2^6 = -\frac{B_3^6 + B_4^6 + B_5^6}{C^2},$$

$$A_3^2 = -\frac{A_4^2}{C^2}, \qquad B_3^2 = -\frac{B_4^2}{C^2},$$

$$A_3^3 = -\frac{A_4^3}{C^2}, \qquad B_3^3 = -\frac{B_4^3}{C^2},$$

$$A_3^4 = -\frac{A_4^4}{C^2}, \qquad B_3^4 = -\frac{B_4^4}{C^2}. \tag{19}$$

When the changes made in (19) are introduced in (18), we find the final form of the solutions to be

$$h_1^0(r) = -\frac{1}{C^2}B_1^5 r^{-5} + \frac{1}{C^2}B_1^0 r^{-3} + \frac{2}{3C^2}A_1^5 - \frac{1}{C^2}(A_3^5 + A_4^5)y_3(r) - \frac{1}{C^2}(B_3^5 + B_4^5)y_3^*(r),$$

$$h_1^1(r) = 3\frac{1}{C^2}B_1^1 r^{-5} - \frac{1}{C^2}A_3^1 y_3(r) - \frac{1}{C^2}B_3^1 y_3(r),$$

$$h_2^1(r) = -\frac{1}{C^2}B_1^1r^{-3} - \frac{1}{C^2}A_1^1 - \frac{1}{C^2}A_3^1y_2(r) + A_3^1y_1(r) - \frac{1}{C^2}y_2^*(r) + B_3^1y_1^*(r),$$

$$\begin{aligned} h_2^2(r) = {} & \frac{1}{5}\frac{1}{C^2}B_1^6r^{-5} - \frac{1}{C^2}B_1^2r^{-3} - \frac{1}{C^2}A_2^2 + \frac{1}{5}\frac{1}{C^2}A_1^6r^2 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_3(r) \\ & + \left(-\frac{1}{C^2}A_4^2 + A_3^6\right)y_2(r) + A_4^2y_1(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_3^*(r) \\ & + \left(\frac{1}{C^2}B_4^2 + B_3^6\right)y_2^*(r) + B_4^2y_1^*(r), \end{aligned}$$

$$\begin{aligned} h_1^2(r) = {} & -\frac{1}{C^2}B_1^6r^{-7} + 3\frac{1}{C^2}B_1^2r^{-5} + \frac{2}{5}\frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) - \frac{1}{C^2}A_4^2y_3(r) \\ & - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) - \frac{1}{C^2}B_4^2y_3^*(r), \end{aligned}$$

$$\begin{aligned} h_1^3(r) = {} & -\frac{1}{C^2}B_1^6r^{-7} + \frac{3}{C^2}B_1^3r^{-5} + \frac{2}{5}\frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) - \frac{1}{C^2}A_4^3y_3(r) \\ & - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) - \frac{1}{C^2}B_4^3y_3^*(r), \end{aligned}$$

$$\begin{aligned} h_2^3(r) = {} & \frac{1}{5}\frac{1}{C^2}B_1^6r^{-5} - \frac{1}{C^2}B_1^3r^{-3} - \frac{1}{C^2}A_2^3 + \frac{1}{5}\frac{1}{C^2}A_1^6r^{-2} - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_3(r) \\ & + \left(A_4^6 - \frac{1}{C^2}A_4^3\right)y_2(r) + A_4^3y_1(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_3^*(r) \\ & + \left(B_4^6 - \frac{1}{C^2}B_4^3\right)y_2^*(r) + B_4^3y_1^*(r), \end{aligned}$$

$$\begin{aligned} h_1^4(r) = {} & -\frac{1}{C^2}B_1^6r^{-7} + \frac{3}{C^2}B_1^4r^{-5} + \frac{2}{5}\frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) - \frac{1}{C^2}A_4^4y_3(r) \\ & - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) - \frac{1}{C^2}B_4^4y_3^*(r), \end{aligned}$$

$$\begin{aligned} h_2^4(r) = {} & \frac{1}{5C^2}B_1^6r^{-5} - \frac{1}{C^2}B_1^4r^{-3} - \frac{1}{C^2}A_2^4 + \frac{1}{5}\frac{1}{C^2}A_1^6r^2 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_3(r) \\ & + \left(A_5^6 - \frac{1}{C^2}A_4^4\right)y_2(r) + A_4^4y_1(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_3^*(r) \\ & + \left(B_5^6 - \frac{1}{C^2}B_4^4\right)y_2^*(r) + B_4^4y_1^*(r), \end{aligned}$$

$$h_1^5(r) = 5\frac{1}{C^2}B_1^5r^{-7} - \frac{1}{C^2}(A_3^5 + A_4^5)y_4(r) - \frac{1}{C^2}(B_3^5 + B_4^5)y_4^*(r),$$

$$h_2^5(r) = -\frac{1}{C^2}B_1^5r^{-5} - \frac{1}{C^2}(A_3^5 + A_4^5)y_3(r) + A_3^5y_2(r) - \frac{1}{C^2}A_1^5 - \frac{1}{C^2}(B_3^5 + B_4^5)y_3^*(r)$$

$$+ B_3^5y_2^*(r),$$

$$h_3^5(r) = -\frac{1}{C^2}B_1^5r^{-5} - \frac{1}{C^2}A_1^5 - \frac{1}{C^2}(A_3^5 + A_4^5)y_3(r) + A_4^5y_2(r) - \frac{1}{C^2}(B_3^5 + B_4^5)y_3^*(r)$$

$$+ B_4^5y_2^*(r),$$

$$h_1^6(r) = \frac{7}{C^2}B_1^6r^{-9} - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_5(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_5^*(r),$$

$$h_2^6(r) = -\frac{1}{C^2}B_1^6r^{-7} - \frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) + A_4^6y_3(r)$$

$$-\frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) + B_3^6y_3^*(r),$$

$$h_3^6(r) = -\frac{1}{C^2}B_1^6r^{-7} - \frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) + A_3^6y_3(r)$$

$$-\frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) + B_4^6y_3^*(r),$$

$$h_4^6(r) = -\frac{1}{C^2}B_1^6r^{-7} - \frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) + A_5^6y_3(r)$$

$$-\frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) + B_5^6y_3^*(r). \tag{20}$$

In the above equations, the functions y_i and y_i^* ($i = 1, 2, 3, \dots$) are defined through equation (15). We remark that constraints (19) do not suggest any changes in the form of solution (7) for the pressure.

3. Hydrodynamic force on a porous sphere

As an application of the method described in the previous section, we now consider the flow of a viscous fluid past a porous sphere. A solution to this problem when the flow, both inside and outside of the sphere, is approximated by Stokes' equation has been considered by Lenov [7]. It is, however, now believed that, in order to describe more accurately the flow inside the porous sphere, Stokes' equation should be replaced by Brinkman's equation. Theoretical justification supporting these views has been proposed by Howells [2], Hinch [3] and many others. On the other hand, the indiscriminate use of this equation, for all situations, has also been questioned. Here we consider the fluid outside the sphere to be a Stokesian fluid with undisturbed velocity U far from the sphere. Inside the sphere we assume that the average velocity and the pressure satisfy the Brinkman equation (2). With the boundary conditions that both the velocity and surface forces be continuous across the surface of the porous sphere we determine the velocity and pressure distributions both inside and outside of the sphere. We then use these quantities to calculate the drag on the sphere.

For the Stokes flow outside the sphere we use the solution obtained in our previous paper [5] and select the Cartesian-tensor form of the boundary conditions as

$$p_\infty = a_{ii}, \quad u_{l\infty} = \varepsilon_{ljk} a_{kj} = U\mathbf{e}_3. \tag{21}$$

It then follows that (c.f. eqs. (20), (21) and (22) of [5])

$$p^{(o)} = p_\infty + A_1^1 r^{-3} U x_3, \tag{22}$$

$$u_l^{(o)} = (A_3^1 r^{-5} + \tfrac{1}{2} A_1^1 r^{-3}) U x_3 x_l + (1 - \tfrac{1}{3} A_3^1 r^{-3} + \tfrac{1}{2} A_1^1 r^{-1}) U \delta_{l3}, \quad (l = 1, 2, 3). \tag{23}$$

where we have added the superscript (o) to denote quantities *outside* of the porous sphere. For the fluid *inside* the sphere we will use the superscript (i). For the flow inside the porous sphere we select

$$p^{(i)} = \bar{A}_1^1 U x_3, \tag{24}$$

$$u_l^{(i)} = -\frac{1}{C^2} \bar{A}_3^1 U x_3 x_l + \left[-\frac{1}{C^2} \bar{A}_1^1 - \frac{1}{C^2} \bar{A}_3^1 y_2(r) + \bar{A}_3^1 y_1(r) \right] U \delta_{l3}, \quad (l = 1, 2, 3). \tag{25}$$

On redefining the four constants appearing in the above equation as

$$A = A_3^1, \quad B = A_1^1, \quad D = \bar{A}_1^1, \quad E = \bar{A}_3^1, \tag{26}$$

we can write the expressions in component form:

$$p^{(o)} = B r^{-3} U x_3,$$

$$u_r^{(o)} = (\tfrac{2}{3} A r^{-3} + B r^{-1} + 1) U \cos\theta,$$

$$u_\theta^{(o)} = (\tfrac{1}{3} A r^{-3} - \tfrac{1}{2} B r^{-1} - 1) U \sin\theta,$$

$$u_\phi^{(o)} = 0,$$

$$p^{(i)} = D U x_3,$$

$$u_r^{(i)} = \left[-\frac{1}{C^2} D + \frac{2}{C^2} E y_2(r) \right] U \cos\theta,$$

$$u_\theta^{(i)} = \left[\frac{1}{C^2} D + \frac{1}{C^2} E y_2(r) - E y_1(r) \right] U \sin\theta,$$

$$u_\phi^{(i)} = 0. \tag{27}$$

On applying the boundary conditions that the velocity and the stress component be continuous across the surface of the porous sphere ($r = a$) we get

$$\tfrac{2}{3} A + Ba^2 + a^3 = -\frac{1}{C^2} Da^3 + \frac{2}{C^2} Ea^3 y_2(a),$$

$$\tfrac{1}{3} A - \tfrac{1}{2} Ba^2 - a^3 = \frac{1}{C^2} Da^3 + \frac{1}{C^2} Ea^3 y_2(a) - Ea^3 y_1(a),$$

$$-4A - 3Ba^2 = -Da^5 + \frac{4}{C^2} Ea^5 y_3(a),$$

$$-\tfrac{5}{3} A - \tfrac{1}{2} Ba^2 - a^3 = \frac{1}{C^2} Da^3 + \frac{1}{C^2} Ea^5 y_3(a) - \frac{2}{C^2} Ea^3 y_2(a) - Ea^5 y_1(a). \tag{28}$$

On solving this system of equations we obtain

$$A = \left[\frac{3}{C^2} a^3 y_2(a) + \tfrac{1}{2} a^5 y_2(a) - a^3 y_1(a) \right] E,$$

$$B = -a^3 y_2(a) E,$$

$$D = -y_2(a) E,$$

$$E = 3a^3 \left[\frac{3}{C^2} a^3 y_2(a) + 2a^5 y_2(a) + 2a^3 y_1(a) \right]^{-1}. \tag{29}$$

We note that in writing (29) we have also used the result that $a^5 y_3(a) + 3a^3 y_2(a) = C^2 a^2 y_1(a)$. Furthermore, we also want to emphasize that a variation in the boundary conditions will result in the different values of the constants. When constants determined in (29) are employed in (27), we get the complete expressions for the pressure and the velocity distribution.

In order to determine the drag on the porous sphere, which will be directed along the symmetrical axis, we need to calculate

$$D = 2\pi a^2 \int_0^\pi (t_{rr} \cos\theta - t_{r\theta} \sin\theta) \sin\theta \, d\theta. \tag{30}$$

On calculating the expressions for the stresses in spherical polar coordinates, by using (26), we find

$$D = -4\pi\mu UB \tag{31}$$

where

$$B^{-1} = -\frac{E^{-1}}{a^3 y_2(a)} = -\frac{1}{3a^6 y_2(a)}\left[\frac{3}{C^2} a^3 y_2(a) + 2a^5 y_2(a) + 2a^3 y_1(a)\right]. \tag{32}$$

Equations (31) and (32) thus give

$$D = 12a^6 \pi\mu U\left[\frac{3}{C^2} a^3 + 2a^5 + 2a^3 \frac{y_1(a)}{y_2(a)}\right]^{-1}. \tag{33}$$

Clearly, when $k \to 0$, i.e., $C \to \infty$, the above reduces to the classical value determined by Stokes (since in that case the porous sphere behaves like a solid sphere and $u_i^{(i)} = 0$). For nonzero values of k, it is found that the drag in a porous sphere is lower than that of a solid sphere and that it decreases when k increases.

It is of some interest to express asymptotic values of the drag D in equation (33) for low and high values of the permeability parameter, $k(=1/C^2)$. For large permeability we can write $y_1(a)/y_2(a)$ in (33) as

$$\begin{aligned}\frac{y_1(a)}{y_2(a)} &= \left[\frac{1}{a}\sinh(Ca)\right]\Big/\left[C\frac{\cosh(Ca)}{a^2} - \frac{\sinh(Ca)}{a^3}\right] \\ &= \left[\frac{C}{a}\coth(Ca) - \frac{1}{a^2}\right]^{-1} \approx \frac{3}{C^2} + \frac{3a^2}{15} = 3\left(k + \frac{a^2}{15}\right).\end{aligned} \tag{34}$$

Subsitution of (34) in (33) and further simplification gives

$$D = 4\mu\frac{\pi Ua}{3}\left[\frac{a^2}{k} - \frac{4}{15}\frac{a^4}{k^2} + O\left(\frac{a^6}{k^3}\right)\right]. \tag{35}$$

For low permeability we can approximate $y_1(a)/y_2(a)$ by

$$\frac{y_1(a)}{y_2(a)} \approx \frac{a}{c} = a\sqrt{k},$$

and in this case (33) gives

$$D = 6\mu\pi Ua\left[1 - \frac{\sqrt{k}}{a} + O\left(\frac{k}{a^2}\right)\right]. \tag{36}$$

We point out that both the above expressions agree with the results of Higdon and Kojima [6] who had derived them by solving integral equations.

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References

1. H.C. Brinkman, A calculation of the viscous forces exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res.* A1 (1947) 27–34.
2. I.D. Howells, Drag due to motion of a Newtonian fluid through a sparse random array of small fixed rigid objects, *J. Fluid Mech.* 64 (1974) 449–475.
3. E.J. Hinch, An averaged-equation approach to particle interactions in a fluid suspension, *J. Fluid Mech.* 83 (1977) 695–720.
4. K. Matsumoto and A. Suganuma, Settling velocity of a permeable model floc, *Chem. Engng. Sci.* 32 (1977) 445–447.
5. R. Niefer and P.N. Kaloni, A Cartesian-tensor solution of the viscous creeping motion equation, *J. Engng. Math.* 20 (1986) 189–201.
6. J.J.L. Higdon and M. Kojima, On the calculation of Stokes flow past porous particles, *Int. J. Multiphase Flow* 7 (1981) 719–727.
7. A.I. Lenov, The slow stationary flow of a viscous fluid about a porous sphere, *P.M.M.* 26 (1962) 564–566.